The Fascination of Tiling

Doris Schattschneider

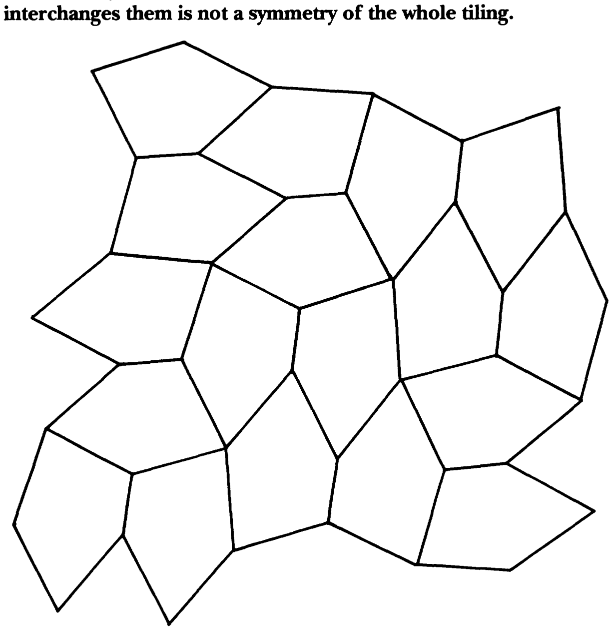
ABSTRACT

Mosaics that cover surfaces have long been of interest to designers and artists. Recently, however, mathematicians have turned their attention to these visual displays and found them a fascinating source of interesting problems, many of which are still unsolved.

Dutch artist M. C. Escher (1898–1972) often described regular divisions of the plane as "the richest source of inspiration I have ever struck" [1]. Interlocking shapes displayed in majolica tile, inlaid wood, brickwork, carved stucco, stone pavement, sewn patchwork or printed fabric hold a special fascination for many people that goes far beyond the aesthetic pleasure that these patterns provide. Tiling can serve as a paradigm: it is wholly visual (and perhaps seems to be purely in the provenance of design) yet incredibly rich as a source of mathematical questions. Many of these questions have implications for those who (like Escher) design intricate and unusual tilings. Other questions probe the limits of possibilities for tilings, investigate the structure of tilings, aim to produce methods of classification or link tilings to physical structures in nature or to invented mathematical structures.

Mathematicians seek to know how tiles can pave surfaces—not just the Euclidean (flat) plane, but also the hyperbolic plane and the surfaces of three-dimensional objects such as spheres, tori (doughnuts) or Möbius bands [2]. They also investigate tilings on surfaces impossible to fully represent in our three-dimensional world, such as a Klein bottle or surfaces in higher dimensions. And tiling problems are not only restricted to surfaces—there are marvelous problems that ask questions about how three-dimensional 'tiles' (such as polyhedra) can pack space, or how higher-dimensional spaces can be packed with tiles.

Fig. 1. A tiling by a convex pentagon discovered by Marjorie Rice. Each pentagon is joined to its mirror-image, yet the reflection that interchanges them is not a symmetry of the whole tiling.



Most of these problems are difficult, and most are unsolved, but people are working on them. If we restrict ourselves to questions about tiling the Euclidean plane, the subject is still very rich and far from complete. In this article, I will highlight some of the different kinds of mathematical questions that can be posed with regard to tilings of the Euclidean plane. Some have been answered fully, others only partially, and some not at all. Although mathematicians and scientists have investigated these questions (and still do), many are accessible to those with little or no formal mathematical training. Playing with the possibilities can provide much pleasure and may even lead to important discoveries.

A *tiling* is a covering of the plane by closed shapes that is without gaps or overlaps; two other synonymous terms are *tessellation* and *parquetry*. Of course, a real tiling has a small space between adjacent tiles that is filled with cement, glue or just plain dirt; mathematically, this space is treated as a heavy outline of tiles that (in theory, at least) fit perfectly together. The fitting together of shapes to fill an area is a pleasurable challenge—play with colored polygon shapes or the completion of a jigsaw puzzle are examples in which the shapes are given, and we expect them to fit together in at least one way. Simple polygon shapes produced for children's explorations of tilings are known to fit together in many different ways; jigsaw-puzzle shapes are guaranteed to fit together because they have been simultaneously cut with a die whose cutting edge outlines a tiling of a rectangle equal in size to the completed puzzle.

But what if we are presented with a box of tiles with no guarantee that they will fit together? Can we predict if a region can be tiled with them? [3] If we allow an infinite number of copies of the tiles, can we predict if we can tile the whole plane with them? Of course, if none of the pieces can fit snugly against each other (such as circular discs, or tiles with indentations that do not match their protrusions), it is easy to decide that the tiles will not fit together to fill

Doris Schattschneider (mathematician), Moravian College, Bethlehem, PA 18018, U.S.A.

Received 17 January 1991.

0024-094X/92 $5.00+0.00

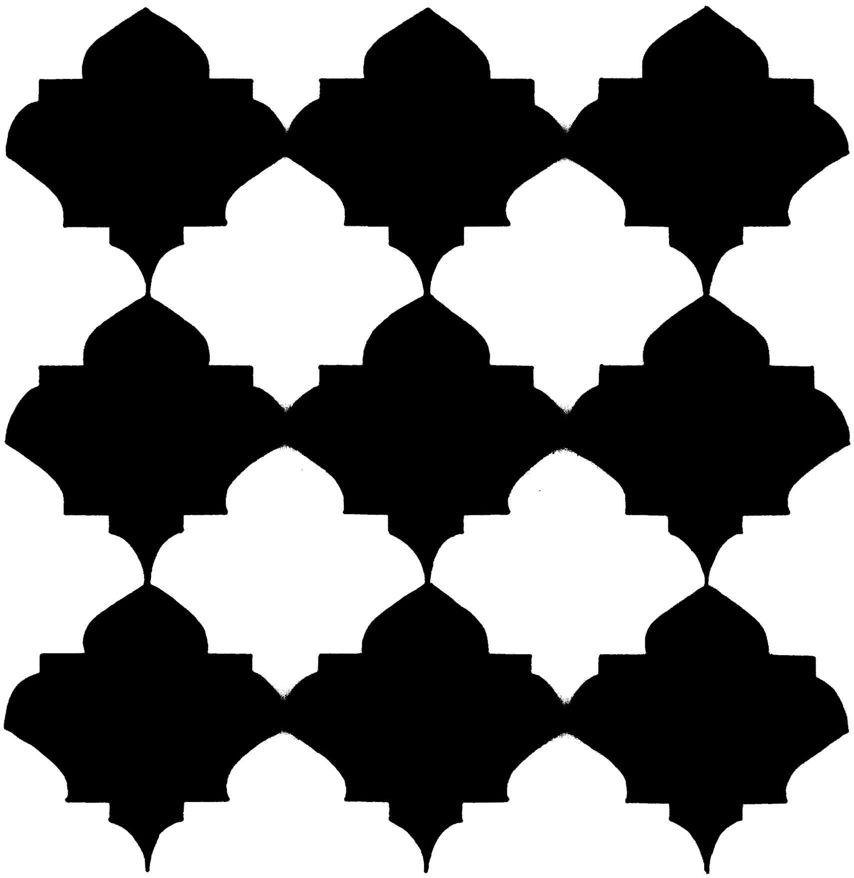


Fig. 2. Edges of these tiles fit together in only one way; this determines a unique tiling of the plane. Note that the bilateral symmetry of the tile induces reflection symmetry into the whole tiling. The alternate black-white coloring produces a counterchange pattern.

out a region. But if there are many 'fits' of edges, can we decide? These basic questions about tiling have a simple (though perhaps unsatisfying) answer: there is no test or algorithm that will show if an arbitrary set of tiles will tile a region equal to their collective area (or the whole plane, if their collective area is infinite). Even when all tiles in the set are copies of a single shape, the answer is the same; this simply means that tiling is not predictable. That is part of the fascination. Mathematicians (and artisans) know there is a rich variety of shapes that *do* tile the plane, and so they pose questions about tiles and tilings with special properties [4]. Answers to some of these questions can be surprisingly useful.

The simplest tilings are those with tiles that are all copies of a single shape. Called monohedral tilings, these have been the most thoroughly investigated by mathematicians, yet there are still many unanswered questions. When we discuss monohedral tilings, we can ask many questions about the single shape (the 'prototile') whose copies fill out the plane. Perhaps the simplest tile shape is a convex polygon, so we first ask which convex polygons can tile the plane. (In more precise mathematical phrasing we ask which convex polygons can be prototiles in a monohedral tiling.) It is easy to show that any triangle can tile the plane, as can any quadrilateral (even nonconvex). But only certain convex pentagons can (regular pentagons, for instance, cannot), and certain hexagons can. It can also be shown that no convex polygons having seven or more sides can tile the plane [5]. The discovery of exactly which convex hexagons can tile the plane (there are exactly three types; they are described by conditions on their sides and their angles) is credited to K. Reinhardt, whose 1918 thesis contains the solution to this problem. The discovery of convex pentagons that can tile the plane is a saga that spans many years and involves discoveries by mathematicians Reinhardt (in 1918) and R. Kershner (in 1968), as well as discoveries by 'amateurs' Richard James (in 1975) and Marjorie Rice (in 1976–1980) and graduate student Rolf Stein (in 1985) (see Fig. 1). Although at least three times in the saga it was claimed that the list of pentagons that tile the plane was complete (and then more were found), there is not yet definitive proof that the known 14 types are all that are possible [6]. Martin Gardner's article "Tiling With Convex Polygons" [7] gives a good overview of the story, and the article "In Praise of Amateurs" [8] tells the story of how Rice pursued the problem after reading about it in Gardner's column in *Scientific American*. This article is a testimony not only to her perseverance, but also to her ingenuity in investigating the problem.

The question of *whether* a shape can tile the plane is inseparable from the question of *how* a shape can tile the plane. Almost all mathematical questions (as well as design questions) that concern tiling by a single shape must take into account the possible ways in which a tile can surround itself. In the simplest case in which a tiling is possible, there is only one way that the tile can be surrounded, and this is the only way that the congruent pieces can continue to fit together to tile the whole plane. This kind of tile is easy to produce and may be the most desirable for those who want a simple and totally predictable job of filling the plane (Fig. 2). But when there is more than one way for congruent pieces to fit together, the possibilities for tiling become very interesting and pose many questions. If copies of a tile can fill out a patch of the plane (even a very large one), can the tiling be continued to fill the plane? If the whole plane can be filled with copies of a tile, is that tiling unique? If not, how many other different tilings with that tile are possible? Although sometimes these questions can be answered, they can be extremely difficult to answer for certain tiles; there is no general algorithm or test that can be applied to any tile [9].

A cross made from five squares is an innocuously simple tile that can fill out any size patch of the plane, but for which there is essentially only one way to fill out the whole plane. If just one cross is wrongly placed, the tiling cannot be completed (Fig. 3). The discovery of a tile for which there is a unique tiling brings special pleasure—especially if the tile is one for which there are many 'false starts' in which the tile fills out a patch and then no additional pieces can be added. Roger Penrose has devised some ingenious tiles with simple shapes (using portions of the boundary of a regular hexagon) so that copies of a single tile 'fit' in a myriad of ways but there is only one way to completely tile the plane with them. He presented a box of wooden cutouts of one such tile (Fig. 4) to Escher in 1962 and challenged him to solve the puzzle. Not only did Escher solve the puzzle (finding the unique way to fit the tiles so the pattern could be continued to fill the plane), but he made his own version, a 'ghost' based on Penrose's geometric piece, and made a color drawing of the interlocked creatures [10].

For specific tiles, simply trying to fit their copies together may yield a tiling, but this approach rarely gives insight into the properties of large classes of tilings. For mathematicians, it is natural to try to discover constraints on tilings, to describe as precisely as possible various types or classes of tilings that satisfy certain properties and fine-tune definitions to rule out anomalous exceptions to what seems to be generally true. It is perhaps surprising that there are geometric constraints that must be obeyed by any 'ordinary' tiling of the plane, whether the tiles are all alike, some different, or all different. One such constraint is that for any tiling of the plane by convex polygons, there must be at least one tile that has six or fewer vertices. In fact, even if the tiles are not convex polygons, a much stronger condition usually holds: the tiling must contain an infinite number of tiles with at most six vertices (a vertex of a tiling is a point where three or more tiles meet) [11]. It is interesting to note here

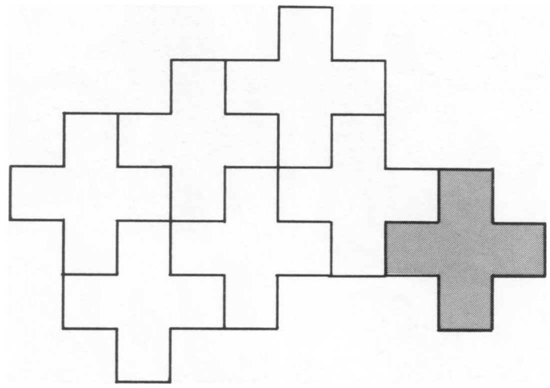


Fig. 3. There is essentially only one way to tile the whole plane by this cross tile: the continued arrangement shown for the white crosses, or the mirror image of this arrangement. If just one tile is wrongly placed (shaded cross), the tiling cannot be continued to fill the plane. Although each cross tile has four axes of reflection symmetry, there is no reflection symmetry in the plane tiling.

Fig. 4. A tile devised by Roger Penrose that can only tile the plane in one way. A tile and its mirror image (shaded) form a unit that can be rotated repeatedly 60° to fill out a snowflake shape; this patch of 12 tiles can fill the plane by translations. The tiling is nonisohedral since no symmetry of the tiling can map a tile to its mirror image.

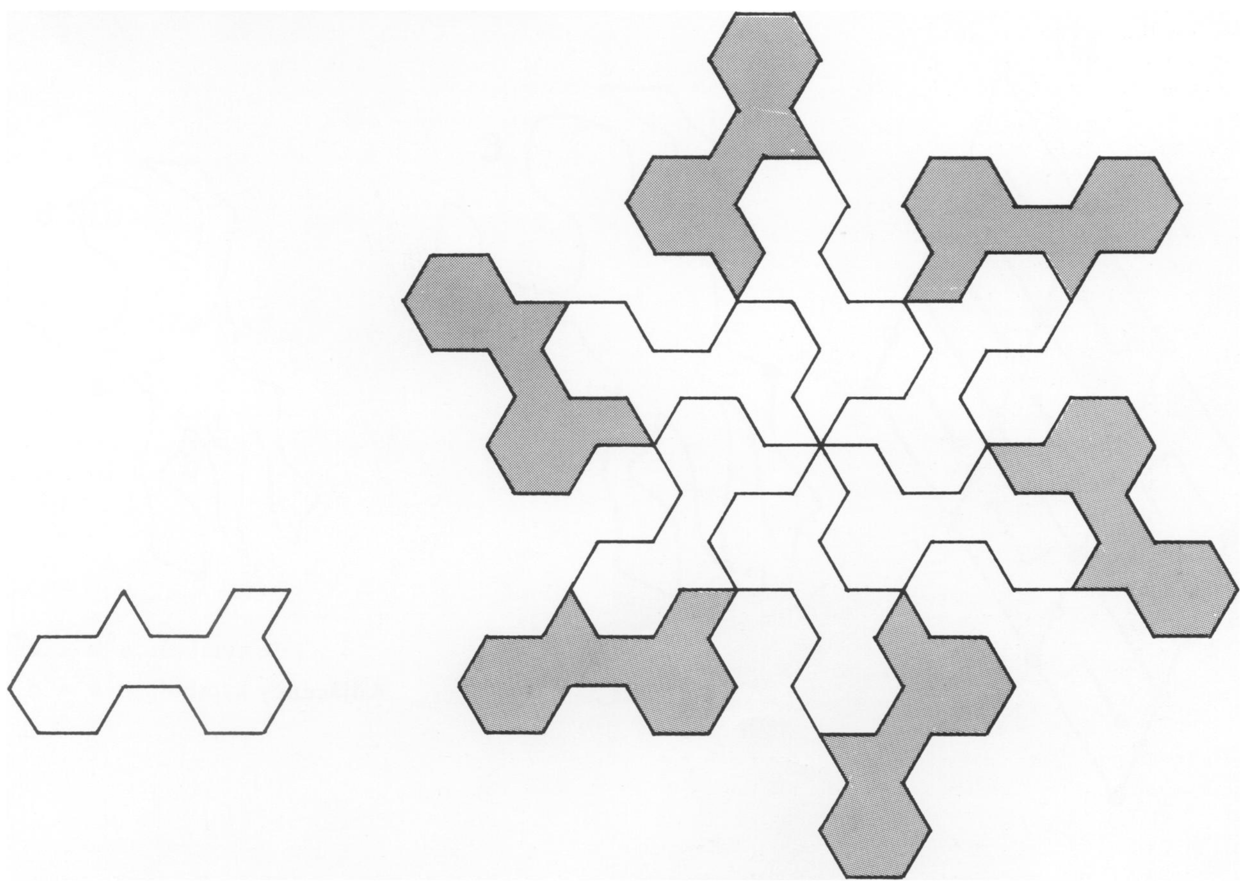
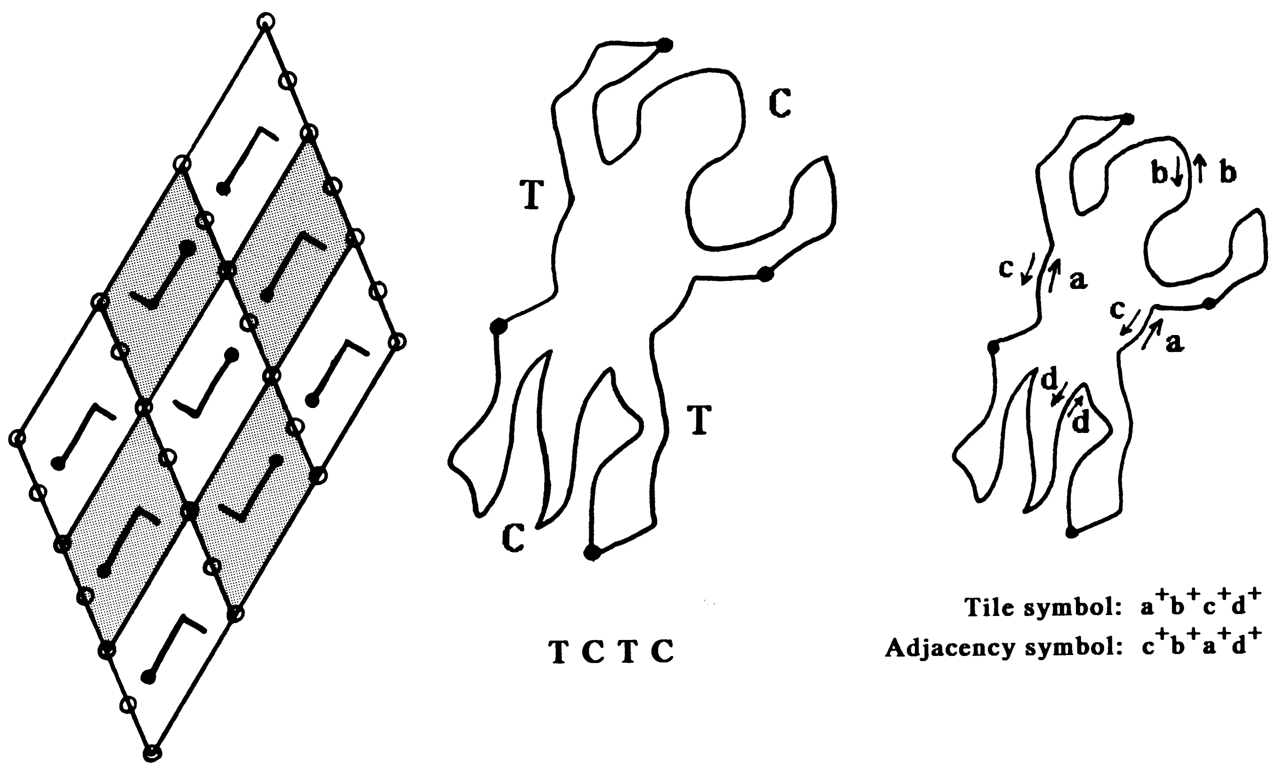


Fig. 5. M. C. Escher, *Symmetry Drawing No. 75*, india ink, pencil, black-and-white paint, 223 × 206 mm, 1949. (© 1991 M. C. Escher Heirs, Cordon Art, Baarn, Holland) (top left) One of Escher's isohedral tilings by lizards. Each lizard in the tiling is surrounded in the same way; this provides a signature that determines the tiling. Three ways of depicting this signature—(bottom left) by Escher, (bottom center) by Heesch and (bottom right) by Grünbaum and Shephard—are illustrated.



that although one might expect constraints on plane tilings to easily lead to similar constraints on packings of three-space, that is not often the case. Although a convex polygon that tiles the plane can have no more than six vertices, the upper bound on the number of faces of a convex polyhedron that can pack space is not known. A convex space-filler with 38 faces found in 1981 by Peter Engel holds the current record for the most faces, but there is no proof that this is the largest number of faces possible [12].

Another constraint that is true for any finite tiling of a patch of the plane is Euler's theorem, which says $v + t = e + 1$, where v is the number of vertices in the tiling, t is the number of tiles, and e is the number of edges in the tiling. (An edge is a portion of the common boundary of two tiles and joins two adjacent vertices.) Under appropriate assumptions, this constraint can be extended to a tiling of the whole plane by forming the ratios v/t and e/t for a patch of the tiling, and then taking the limit as the patch grows larger to cover the whole plane [13].

To be able to describe and classify large classes of tilings, it is natural for mathematicians to restrict questions to tilings with special structure and orderliness. The mathematical model for the internal structure of crystals is one in which atoms are aligned in a periodic lattice, so (until very recently) almost all mathematical investigations of tilings have been restricted to those that are periodic. In a periodic tiling, there is always a minimal patch of the tiling that fills the whole tiling by translating it again and again in two different directions. (One can first fill out an infinite strip with the patch, then translate the strip to fill out the plane.)

In their quest to find out how many different kinds of periodic tilings there are, investigators concentrated on the global order of these tilings. Mathematicians use geometric transformations of the plane that preserve shape—isometries—to describe this global order. These transformations are translation, rotation, reflection and glide-reflection. To analyze a particular periodic tiling, mathematicians seek to learn which isometries transform the tiling so that it will superimpose exactly on itself. Each such geometric transformation is called a *symmetry* of the tiling; the collection of all symmetries of a tiling is called the *symmetry group* of the tiling. Symmetry groups are not just collections of geometric transformations but also have algebraic structure. Periodicity of a tiling severely limits the possibilities for symmetries other than translations; there are only 17 different types of symmetry groups of periodic tilings. Classification by symmetry group provides a convenient and straightforward method of categorizing periodic tilings. Symmetry groups also provide a way of generating periodic tilings (as well as periodic patterns) since each group is generated by just a handful of isometries. If one draws any asymmetric figure in the plane, and the isometries that generate one of the 17 groups act repeatedly on the figure, a periodic design will result that has the prescribed symmetry group [14]. This means that computer programs can be written that will automatically generate periodic tilings or patterns of any of the designated 17 types [15].

The problem with classification of tilings by symmetry groups is that it gives no information whatsoever about the shapes of the tiles, about whether all tiles are of the same shape or of several different shapes, nor any information about how an individual tile is surrounded by other tiles. To test to see if two periodic tilings are 'the same' under this classsification is like putting the same mask on each one—the mask shows reflection axes, rotation centers and glide axes for the whole tiling. If the mask (scaled separately in two directions if necessary) fits both, they are the same. We learn no visual information about the design of the tiling, only information about its symmetry.

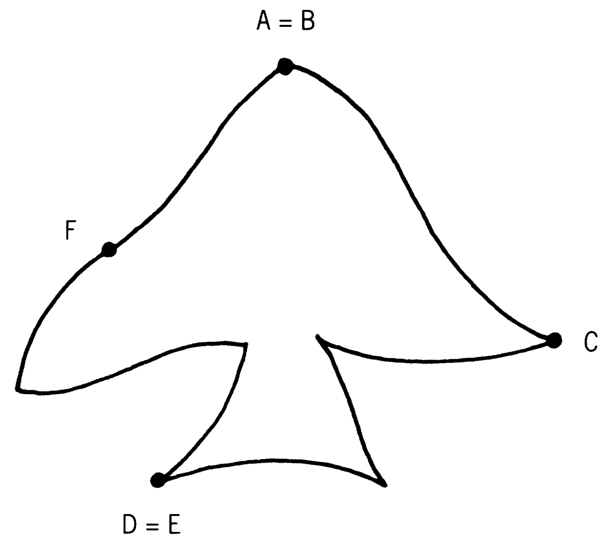


Fig. 6. A Conway Criterion tile can fill the plane by successive 180° rotations. The tile shown here has exactly four centrosymmetric edges; compare to Escher's lizard in Fig. 5, another example that has two centrosymmetric edges joined by two edges that match by translation.

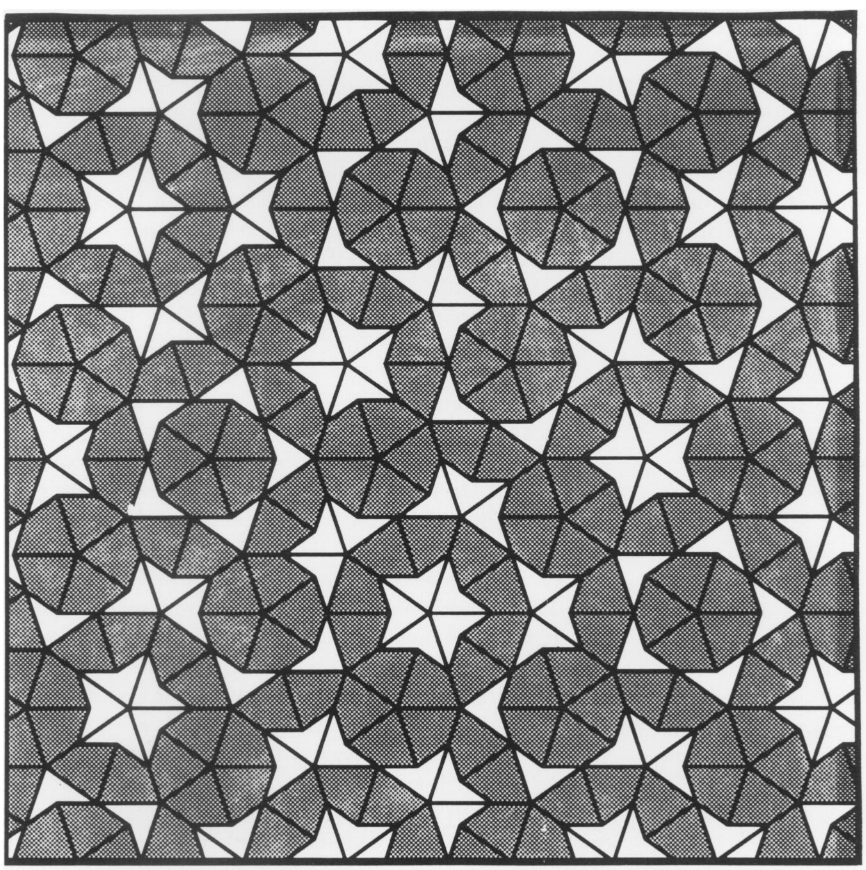
When we consider tilings with all congruent tiles, classification by symmetry groups seems especially inadequate. When all tiles in a tiling are copies of a single shape, portions of the edges of the tile must match, and so a signature of matching can be assigned to the consecutive edges of a tile that describes how those edges match the edges of the tiles that surround it. Merely knowing how one tile can be surrounded is rarely enough to determine a whole tiling (see the cross tile and the Penrose tile in Figs 3 and 4). But for periodic tilings that have the special property that *every* tile is surrounded in the *same* way by its copies, this signature is enough to determine the whole tiling. Escher called such periodic tilings 'regular'; mathematicians call them 'isohedral', or 'tile-transitive'. In 1977 Branko Grünbaum and G. C. Shephard answered the question of how many different isohedral tile types there are—81 distinctly different types. These mathematicians introduced the notion of an 'adjacency symbol' for an isohedral tiling. To produce the symbol, edges of one tile are labeled consecutively with letters, and then the letters corresponding to the shared edges of the tiles that surround the first tile are listed in sequence. It is interesting to note that, earlier, the mathematician H. Heesch and the graphic artist Escher had each (independently) developed classification systems for some of these tilings, based on a 'local signature' that described how each tile was related by isometries to its surrounding tiles. These isometries not only give information on how the tiles fit, but also constrain the allowable shapes for the edges of the tile. Figure 5 shows one of Escher's isohedral tilings by a single lizard and illustrates how it would be described by Escher, by Heesch and by Grünbaum and Shephard [16]. Escher's own schematic diagram (in which a parallelogram marked with a hook represents a lizard) shows half-turn centers as small circles and indicates by the orientation of the hooks how a given tile is surrounded by other tiles. Each lizard tile has four vertices, marked by black dots in the

outline of the tile. Heesch's labeling of a tile shows that two edges match by translation (T), and two are centrosymmetric (C)—a half-turn about the midpoint of either of these edges matches the edge to itself. The adjacency symbol of Grünbaum and Shephard shows how directed edges consecutively labeled (a, b, c, d) fit against edges of adjacent tiles with the same labeling; the + indicates same orientation.

This description of local (rather than global) structure is a natural one for an artisan who wishes to design a tile that will fill the plane isohedrally. By reading the coded shapes in Heesch's chart, or in Grünbaum and Shephard's diagrams of isohedral tilings, found in their book *Tilings and Patterns,* one can easily create original isohedral tiles. A particularly simple recipe to follow is the Conway Criterion, which produces tiles that will fill the plane simply by applying 180° rotations to some vertices and the centers of some tile edges. The boundary of a Conway Criterion tile has six consecutively labeled vertices (A, B, C, D, E, F) and satisfies the condition that the edge from A to B matches the edge from E to D by translation, and the remaining edges BC, CD, EF and FA are centrosymmetric. Some of the vertices can coincide; at least three distinct vertices are necessary. Escher's lizard in Fig. 5 fits this recipe; another tile that satisfies the criterion is shown in Fig. 6 [17]. A computer program called Mosedit has been developed at the University of Montreal by J. Baracs and N. Chourot to assist in the creation of isohedral tiles, based on the classification of Grünbaum and Shephard. The user chooses a tile type and then can design an original shape of that type, with the computer program forcing the outline of the tile to obey the necessary constraints.

In his classification of regular tilings based on the relation of each tile to its adjacent copies, Escher did not consider reflection symmetries; this was probably because his creature-shaped tiles rarely had straight edges that would allow them to reflect into an adjacent tile. Yet several of his tilings have reflection symmetry, introduced by the use of a tile that is bilaterally symmetric (such as a fish, an angel or a bat). The tiling in Fig. 2 shows such induced reflection symmetry. This raises the interesting question of when the symmetry of an individual tile would necessarily introduce similar symmetry into an isohedral tiling. Certainly not always: the cross tile in Fig. 3 has four reflection axes that intersect in a four-fold center of rotation, yet the unique plane tiling with this cross has no reflection symmetry (it does have four-fold rotation symmetry). The cross tile is an example of a *hypersymmetric* tile—it possesses additional symmetry that is not present in any tiling by the tile. Some of Escher's birds and fish are hypersymmetric or nearly so; he noted this on his drawings by saying that they were 'apparently symmetric'. The question of when a symmetric tile introduces symmetry into an isohedral tiling (and when it does not) is an open one. In fact, both the question of for which isohedral types do hypersymmetric tiles exist and the question of what extra symmetries are possible for a tile of a given isohedral type are not yet answered [18].

It is easy to produce nonisohedral tilings by a single tile (even something as simple as a triangle will work), that is, not every tile is surrounded in exactly the same way. But are there tiles that can fill the plane *only* in a nonisohedral way? In 1900, at the International Congress of Mathematicians in Paris, David Hilbert posed a list of important mathematical problems, and among these was a suggestion that he thought the answer was probably 'no' [19]. Yet the answer is yes; Heesch provided the first examples of such tiles in 1935. These were nonconvex polygons with interlocking teeth. Penrose's tile in Fig. 4 is another example of such a tile—this tile can only fill the plane if the shapes are surrounded in



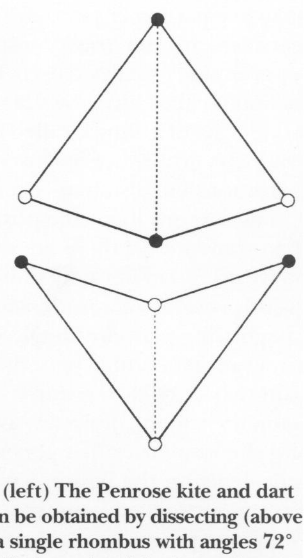


Fig. 7. (left) The Penrose kite and dart tiles can be obtained by dissecting (above right) a single rhombus with angles 72° and 144°; the ratio of the long edges to the short edges of the tiles is the golden number $\tau = (1 + \sqrt{5})/2$. Vertices of the two tiles are colored as shown; only vertices of the same color are allowed to match. Although there is an infinite number of tilings that follow these rules, every one is aperiodic. The drawings here were produced by Stan Wagon, using the computer software program *Mathematica* [33].

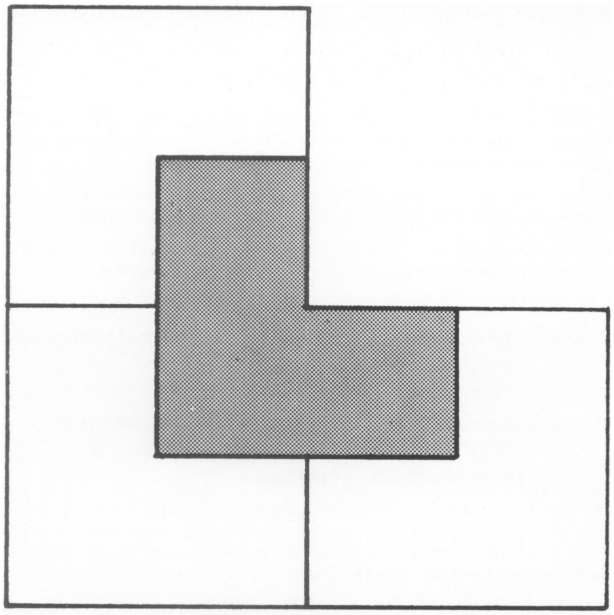
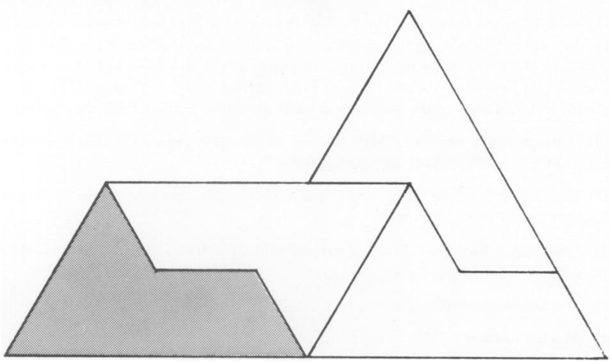


Fig. 8. Two 'rep-tiles' (above and below), each able to fill a similar shape that has scale twice that of the original tile.



two distinctly different ways. Two of the tiles together can form a unit that tiles isohedrally—this unit fills out a patch that fills the plane by translations. In 1968, Kershner published even more surprising news: he had found three different types of *convex* pentagons that could tile the plane, but only nonisohedrally. Until then, it was generally believed that all convex polygons that could tile the plane must be able to do so isohedrally. The later discoveries of other types of convex pentagons that tile the plane provided still more examples of polygons that can only tile nonisohedrally (see Fig. 1) [20]. So far, for every tile that has been found that can only tile nonisohedrally, a unit can be formed from a few copies of the tile so that the unit *can* tile isohedrally. It is not known if this phenomenon must be true for every tile that can only fill the plane in a nonisohedral way.

The requirement of a high degree of some kind of special structure in a tiling makes it more tractable for the mathematician to examine and catalog, and periodic tilings have received the most attention by investigators. But recently, there has been growing interest in understanding *aperiodic* tilings—those formed from sets of tiles whose copies can fill the plane in many different ways, but that can never form a periodic tiling. The most famous of these tilings was discovered by Penrose about 15 years ago, and is formed (in one version) by copies of two shapes of tile—a kite and a dart—that have strict rules concerning how their edges can match. Locally, the tilings have a great deal of symmetry (some portions have the symmetry of a regular pentagon), and patches of a tiling repeat an infinite number of times throughout the tiling. Yet this repetition is not periodic, not stamped out by repeated translation (see Fig. 7). The attempt to understand how such tilings can arise—how certain shapes, together with matching rules, can only tile aperiodically—is just one of the challenging problems in this field [21].

Another area in which there has been much recent investigation concerns colored tilings—the colors of the tiles can be used as a code for information or to make individual tiles recognizable through contrast or purely for the purpose of a pleasing design. Just as symmetry groups can be used to classify tilings, they can also be used to classify symmetric colorings of tilings. These classifications are used by crystallographers and can also be useful to classify colored periodic designs, particularly the frequently encountered two-color 'counterchange' designs in which the same motif occurs in black and in white, such as the ones in Figs 2 and 5 [22]. Escher made many such two-color tilings; they were a principal device in many of his prints in which the same motif (repeated alternating in two colors) serves both as figure and as ground. Yet symmetry groups do not seem to be a wholly adequate tool to describe and classify colored tilings. There are many examples of very orderly arrangements of colors in certain tilings, yet the classification by symmetry groups completely fails to recognize this orderliness. This particular area is one in which new ideas are sought in order to understand the possibilities for integration of two kinds of orderliness: the arrangement of tiles and the arrangement of colors.

Although most of the problems I have discussed concern tilings by a single tile, there is much interest in tilings by more than one type of tile. The same kinds of classification and symmetry questions mathematicians ask about isohedral tilings can be posed for tilings by two or more types of tiles. Some of these (such as how many 'two-isohedral' tilings there are) have recently been answered, but there are more open questions than answered ones. Escher's original approaches for creating tilings by two different tiles that contrast in both form and in color [23] and Escher's investigations into how to derive new tilings from given tilings by a process that he called 'transition' have only recently been published [24]. There is no doubt that mathematicians will find new questions posed by his work.

Tiling problems are a favorite domain of those who enjoy solving problems and puzzles, and are found in many articles and books on 'recreational' mathematics. Most of these problems do not have straightforward solutions, and those with advanced mathematical backgrounds do not necessarily have an advantage in solving them. Special tiles made by matching the edges of congruent squares (such as the cross tile in Fig. 3) are called polyominoes. This type of tile invites questions such as: Which of these can tile the plane? Which of these can tile a rectangle? Can a particular set of polyomino shapes (such as all those made from five squares) tile a rectangle? Can copies of one shape be put together to form a larger but similar shape? Analogous questions can be asked for tiles formed by matching congruent equilateral triangles or congruent regular hexagons [25]. A single tile that can fill a larger region that has the same shape as the tile has been called a 'rep-tile' by Solomon Golomb; finding rep-tiles (not just among the polyominoes) is an enjoyable challenge [26] (Fig. 8.). This problem can also be posed in the opposite way: Given a particular tile, is it possible to dissect it into congruent pieces that are similar in shape to

the given tile? 'Rep-tiling' has an interesting consequence: it provides a process of tiling the plane that is wholly different than the periodic process of repeated translation. Dubbed 'inflation' by John Conway, this process begins with a single shape, uses a finite number of copies of it to fill out a similar but larger shape, then repeats the process with these larger shapes—as the process continues *ad infinitum,* the tiling must grow to fill the whole plane. This kind of process can describe a way for patches of the known aperiodic tilings to fill the plane, and that is the key to proving their aperiodicity [27].

Historically, interest in tiling has never seemed to flag among designers and artisans. There seems to be no period in which tilings have not been used in both utilitarian and decorative ways; cultures freely borrowed ideas from others and developed their own particular styles in the design of tilings. Yet, until recently, there has been little interest in the subject among mathematicians. With only a few notable exceptions (e.g. Johannes Kepler), mathematicians ignored the subject or relegated it to the category of frivolous, or recreational, mathematics. In the late nineteenth century, questions of crystallographic structure and classification brought some mathematical attention to the subject of tiling, but in the first half of this century, only a handful of mathematicians pursued mathematical questions of tiling.

Today, the 'tiling industry' in mathematics is one of phenomenal growth. The wide variety of difficult and mathematically interesting problems that the subject poses has been brought to the attention of the mathematical community by researchers such as Grünbaum, Shephard, Conway and Penrose. Now many mathematicians and scientists in seemingly unrelated fields of research have discovered that other problems often translate into tiling problems, and so new vigor and insight have enriched the subject. And surprising applications of tiling abound: 'Dirichlet', or 'Voronoï', tilings can illustrate or help solve problems of distribution (of data, supplies, particles or people) [28]; aperiodic tilings are examined for analogies with newly discovered 'quasicrystals'; tilings are investigated to understand structures in nature and to create architectural structures; tilings can be used in teaching to visually illustrate abstract algebraic concepts [29]; tilings are used to 'automatically' generate groups [30]; symmetry classifications of tilings can be used by archeologists to understand cultural styles [31]; tilings are used to understand, simplify or create circuits and other connections; and tilings provide a graphic way of illustrating difficult logical questions of 'undecidability' [32]. And, of course, tiling is fun, a never-ending source of creative diversion. Today, many mathematicians might join Escher in his enthusiastic endorsement, that tiling is the richest source of inspiration ever struck.

References and Notes

1. M. C. Escher, *The Graphic Work of M. C. Escher* (New York: Ballantine Books, 1967) p. 9.

2. For an explanation of how some interesting nonplanar surfaces can be tiled, see Marjorie Senechal, "Escher Designs on Surfaces", in H. S. M. Coxeter, M. Emmer, R. Penrose and M. Teuber, eds., *M. C. Escher: Art and Science* (Amsterdam: North Holland, 1986) pp. 97–122; in the same book, see also Douglas Dunham, "Creating Hyperbolic Escher Patterns", pp. 241–248.

3. Mathematicians use the word *tile* as both noun and verb, just as in common usage: "I tiled my kitchen floor with square tiles."

4. For the most comprehensive source of information on all aspects of tiling, see Branko Grünbaum and G. C. Shephard, *Tilings and Patterns* (New York: Freeman, 1987).

5. Ivan Niven, "Convex Polygons which Cannot Tile the Plane", *American Mathematical Monthly* **85**, No. 10, 785–792 (1978).

6. Doris Schattschneider, "Tiling the Plane with Congruent Pentagons", *Mathematics Magazine* **51**, No. 1, 29–44 (1978). See also *Mathematics Magazine* **58**, No. 5 (1985) p. 308 and cover.

7. Martin Gardner, "Tiling with Convex Polygons," in *Time Travel and Other Mathematical Bewilderments* (New York: Freeman, 1988) pp. 163–176.

8. Doris Schattschneider, "In Praise of Amateurs", in David Klarner, ed., *The Mathematical Gardner* (New York: Wadsworth, 1981) pp. 140–166.

9. See Grünbaum and Shephard [4] section 1.5.

10. Roger Penrose, "Escher and the Visual Representation of Mathematical Ideas", in Coxeter et al. [2] pp. 143–157.

11. See Grünbaum and Shephard [4] sections 3.2 and 3.10.

12. Ludwig Danzer, Branko Grünbaum and G. C. Shephard, "Does Every Type of Polyhedron Tile Three-Space?", *Structural Topology* **8** (1983) pp. 3–14.

13. See Grünbaum and Shephard [4] chapter 3.

14. Doris Schattschneider, "In Black and White: How to Create Perfectly Colored Symmetric Patterns", *Computers and Mathematics with Applications* **12B**, Nos 3/4, 673–695 (1986); Also published in István Hargittai, ed., *Symmetry: Unifying Human Understanding* (New York: Pergamon, 1986) pp. 673–695.

15. Several such programs are described in the Software Review section of this issue.

16. For their classification of isohedral tilings, see Grünbaum and Shephard [4] chapter 6. For Heesch's classification system, see Heinrich Heesch and Otto Kienzle, *Flächenschluss. System der Formen lückenlos aneinanderschliessender Flachteile* (Berlin: Springer, 1963). For the story of Escher's development of his own theory of tiling and classification system, see Doris Schattschneider, *Visions of Symmetry: Notebooks, Periodic Drawings and Related Work of M. C. Escher* (New York: Freeman, 1990). This volume includes color photos of Escher's symmetry drawings, and also a reproduction of Heesch's classification chart.

17. Doris Schattschneider, "Will it Tile? Try the Conway Criterion!", *Mathematics Magazine* **53**, No. 4, 224–233 (1980).

18. Branko Grünbaum, "Mathematical Challenges in Escher's Geometry", in Coxeter et al. [2] pp. 53–68.

19. See Felix E. Browder, ed., *Mathematical Developments Arising from the Hilbert Problems*, 3rd Ed. (Providence, RI: American Mathematical Society, 1979).

20. See Schattschneider [6].

21. Martin Gardner, "Penrose Tiling", and "Penrose Tiling II", in *Penrose Tiles to Trapdoor Ciphers* (New York: Freeman, 1989) pp. 1–29; and Grünbaum and Shephard [4] chapter 10.

22. For detailed information on the symmetry analysis of two-color patterns, see Dorothy Washburn and Donald Crowe, *Symmetries of Culture: Theories and Practice of Plane Pattern Analysis* (Seattle, WA: Univ. of Washington Press, 1988). This volume contains many examples of patterns from archeological sources.

23. Escher's tilings by two tiles that contrast in form and in color (such as birds and fish) invite animation. The movie *M. C. Escher: Symmetry and Space* explores this. See Michele Emmer, "Movies on M. C. Escher and their Mathematical Appeal", in Coxeter et al. [2] pp. 249–262.

24. See Schattschneider [16].

25. See Martin Gardner, "Tiling with Polyominoes, Polyiamonds and Polyhexes", in Gardner [7] pp. 177–187.

26. See Grünbaum and Shephard [4] section 10.1.

27. This process is demonstrated with several pairs of tiles discovered by Penrose, and with a pair of triangle tiles discovered by Raphael Robinson, as described in Grünbaum and Shepard [4] section 10.3.

28. For definitions and illustrations of these tilings, see Marjorie Senechal, *Crystalline Symmetries: An Informal Mathematical Introduction* (Bristol: Adam Hilger, 1990) section 3.3. For references to literature on applications, see Grünbaum and Shephard [4] pp. 265–266.

29. Marjorie Senechal, "The Algebraic Escher," *Structural Topology* **15** (1988) pp. 31–42.

30. William Thurston, "Conway's Tiling Groups", *American Mathematical Monthly* **97** (1990) pp. 757–773.

31. See Washburn and Crowe [23].

32. See Grünbaum and Shepard [4] chapter 11.

33. This tiling is a portion of one that appears in Stan Wagon, *Mathematica in Action* (New York: Freeman, 1991).